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Abstract. The Melnikov method is used to study the transition from a completely integrable classical potential scattering system to a non-integrable one. We investigate the heteroclinic bifurcations and prove the existence of arbitrarily many elliptic and hyperbolic periodic orbits in a bounded scattering region. For selected parameter values some of the analytical predictions are checked by numerical scattering experiments.

1. Introduction

A classical potential scattering system is chaotic if the deflection function or any other convenient property of the final asymptote is discontinuous on a Cantor subset of its domain, which is the set of all incoming asymptotes [1].

Such a type of behaviour can be understood as follows. Within a bounded scattering region an invariant hyperbolic set Λ exists which contains bounded chaotic orbits. The invariant manifolds of this set reach out into the asymptotic region and influence the scattering trajectories. The incoming trajectories feel the existence of the set Λ and the deflection function shows a copy of the complicated Cantor structure. Scattering trajectories coming close to Λ spend a long time inside the interaction region and run alongside localized orbits. Therefore, scattering chaos can be considered as a kind of transient chaos.

As a system parameter is varied, transitions between regular and chaotic behaviour take place also in a scattering process. Recently Bleher *et al* [2] and Ding *et al* [3] studied the generic bifurcation scenarios. Chaotic scattering occurs via a saddle-centre bifurcation with further qualitative changes in the chaotic set resulting from a sequence of homoclinic and heteroclinic intersections or by an abrupt bifurcation to fully developed chaotic scattering. The term *fully developed* chaotic scattering means that all periodic orbits are unstable and there are no KAM surfaces. The abrupt bifurcation arises as the particle energy E decreases from above a critical value E_m, where E_m is one of the maxima of the potential function which is supposed to consist of several hills. Tél [4] has studied this type of bifurcation by using the thermodynamic approach to chaotic processes.

In this paper we investigate another route which is described by the transition from a completely integrable scattering system to a non-integrable one. As usual integrability means the existence of n independent integrals of motion in involution (n is the number of degrees of freedom). Because integrability is not generic the considered route is the exceptional case. Nevertheless, there are a lot of important examples where this transition is realized, and the advantage of using a near-integrable system is given by the fact that one can use some powerful analytical methods to extract important information. For example, the problem of finding transverse heteroclinic orbits can be solved analytically by the Melnikov method [5]. On the other hand, the same problem can be tackled by a computer showing that there are crossing orbits [6].

To simplify matters we choose a four-hill scattering potential depending upon two space variables only. However, we suppose that many results can be extended to the three-dimensional case. The paper is organized as follows: section 2 contains a proof of the existence of the invariant hyperbolic set Λ in the scattering problem. In section 3 the Melnikov method is used to study the bifurcation of smooth resonant tori of the unperturbed system into discrete periodic motions of the perturbed system. Finally, some of the analytical predictions are checked by numerical scattering experiments.

2. Existence of transverse heteroclinic orbits

We consider the following classical potential scattering system. Let F be the function

$$F(x, y) = \frac{x^2+y^2}{2} - \frac{x^4+y^4}{4} + \varepsilon x^2 y^2 \tag{2.1}$$

where $(x, y) \in R^2$ (ε is a small parameter) and let B be the bounded set

$$B \equiv \{(x, y) | F(x, y) > 0\}$$

Then our system is given by the Hamiltonian function

$$H(p_x, p_y, x, y) = \frac{p_x^2 + p_y^2}{2} + V(x, y) \tag{2.2}$$

with the potential

$$V(x, y) \equiv \begin{cases} F(x, y) & \text{for } (x, y) \in B \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 shows a graph of the potential for $\varepsilon = 0.05$. Outside of B there is always a free motion. If the perturbation parameter ε is zero the system is completely integrable,

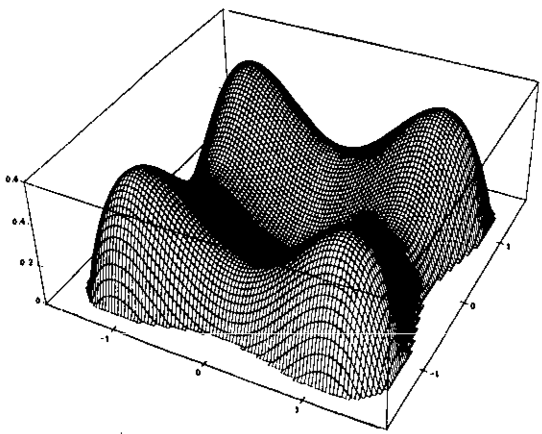

Figure 1. Surface plot of $V(x, y)$ with $\varepsilon = 0.05$.

since the equations of motion split into two independent nonlinear oscillators of the Duffing type for $(x, y) \in B$

$$
\begin{aligned}
\dot{x} &= p_x \qquad & \dot{p}_x &= -x + x^3 \\
\dot{y} &= p_y \qquad & \dot{p}_y &= -y + y^3 .
\end{aligned}
\tag{2.3}
$$

The overdot indicates differentiation with respect to the time variable. From the conservation of energy

$$
\begin{aligned}
H_x &= \frac{p_x^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} = E_x \\
H_y &= \frac{p_y^2}{2} + \frac{y^2}{2} - \frac{y^4}{4} = E_y
\end{aligned}
\tag{2.4}
$$

of the single oscillators the non-existence of crossing orbits for the four-hill potential directly follows for the case $\varepsilon = 0$. Figure 2 illustrates the structure of the phase space of the bounded solutions of the unperturbed system. We shall be interested in the geometrical structure of solutions of the perturbed system ($\varepsilon \neq 0$) lying near the product flow $\Gamma \times \gamma$, where Γ is the heteroclinic solution of the system 1 and γ is one of the periodic orbits of system 2. The case $\gamma \times \gamma$ is discussed in section 3. The corresponding unperturbed solutions are

$$
\begin{aligned}
x(t - t_0^1) &= \pm \tanh \frac{t - t_0^1}{\sqrt{2}} \\
y(t - t_0^2) &= \frac{\sqrt{2} k}{\sqrt{1 + k^2}} \, \mathrm{sn}\left(\frac{t - t_0^2}{\sqrt{1 + k^2}}, k \right)
\end{aligned}
\tag{2.5}
$$

where k is the elliptic modulus which is connected with the energy E_y of the system 2 by $E_y = k^2/(1 + k^2)^2$. t_0^1 and t_0^2 are the initial times of the independent oscillators. The unperturbed solutions (2.5) correspond to values of the energy given by $E_x = 0.25$ and $0 < E_y < 0.25$.

The Melnikov method is an effective tool for the study of the perturbed system. With the unperturbed solutions the so-called Melnikov function has to be calculated, which is a measure of the distance between the stable and unstable manifolds of hyperbolic fixed points in the Poincaré map of the perturbed system. If this function has simple zeros, then there exist transverse homoclinic (heteroclinic) orbits. According to the Smale–Birkhoff homoclinic theorem [5] or the heteroclinic theorem of Bertozzi [7] chaotic motions therefore appear. As a rule the Melnikov method is applicable to nonlinear oscillators with a small time periodic perturbation [5, 8]; however, the case of autonomous perturbations can also be treated (see e.g. [9]). Additionally the

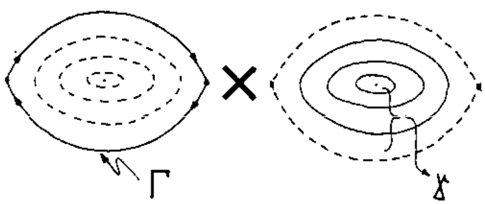

Figure 2. Structure of the phase space of unperturbed bounded solutions.

existence of a Smale horseshoe implies that no analytic integral of motion exists for the perturbed problem, other than the total energy H [10].

We next fix the surface of section

$$\Sigma = \{(x, p_x) | y = 0, p_y > 0\}$$

and consider the Poincaré map $P: \Sigma \to \Sigma$ induced by the solutions of the equations of motion. Note that the map P is piecewise smooth. Moreover, the Melnikov method and the horseshoe construction of Smale and Bertozzi can be extended to mappings of this type [11, 12].

Using the Fourier series representation of the elliptic sine function, the Melnikov integral can be evaluated to give

$$M(\tau_0, k) = \frac{4\pi^3}{(1+k^2)K^2(k)} \sum_{n=0}^{\infty} \frac{\Omega_n^2 \sin(2\Omega_n \tau_0)}{\sinh^2[(2n+1)\pi K'(k)/2K(k)] \sinh(\pi\Omega_n)} \tag{2.6}$$

where $K(k)$ is the complete elliptic integral of the first kind, $K'(k)$ is the complementary elliptic integral, $\tau_0 = (t_0^2 - t_0^1)/\sqrt{2}$ and $\Omega_n = (2n+1)\pi/\sqrt{2(1+k^2)}K$. This function is periodic in τ_0 with period $K\sqrt{2(1+k^2)}$ and has simple zeros. Moreover, a simple calculation yields

$$\max_{\tau_0 \in R} M(\tau_0, k) = \frac{4\pi^3}{(1+k^2)K^2} \sum_{n=0}^{\infty} \frac{(-1)^n \Omega_n^2}{\sinh^2[(2n+1)\pi K'/2K] \sinh(\pi\Omega_n)}. \tag{2.7}$$

The maximum of the Melnikov function characterizes the width of the main stochastic region, i.e. the stochastic layer near the unperturbed separatrix [5].

Figure 3 shows a graph of $\max_{\tau_0 \in R} M$ as a function of E_y, where the dependence $k = k(E_y)$ is used. The results obtained from the heteroclinic Melnikov function are summarized in the following theorem (cf [9]).

Theorem 1. For all $\varepsilon \neq 0$ sufficiently small the scattering system contains horseshoes in its dynamics on every energy level $E \in (0.25, 0.5)$, and hence possesses no analytic second integral. The width of the main stochastic region is given by $d_{\max} = \varepsilon\sqrt{2} \max_{\tau_0 \in R} M(\tau_0, k)$.

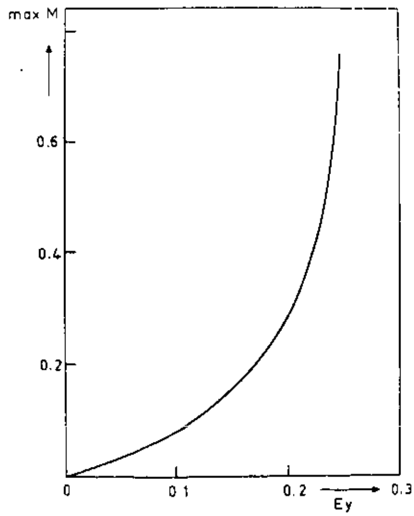

Figure 3. Representation of $\max_{\tau_0 \in R} M$ as a function of E_y.

In order to study the influence of the existence of the invariant hyperbolic set Λ on the scattering process a typical scattering experiment is considered. We choose initial conditions outside of B such that the incoming trajectories are straight lines parallel to the x axis ($p_y=0, p_x=-\sqrt{2E}$). The initial value of the y coordinate is referred to as the impact parameter b.

Because we discuss the near-integrable case it is possible to predict an approximate value of the impact parameter at which the critical behaviour is expected. In a first step we look for trajectories which lie on the stable manifold of the right hyperbolic fixed point on Σ in the unperturbed case $\varepsilon=0$. The corresponding impact parameters b_0 are simply given by $b_0=\pm y_m$, where y_m is the amplitude of the periodic saddle orbit at $x=1$. Using energy conservation we obtain

$$b_0=\pm\sqrt{1-\sqrt{1-4E_y}}=\pm\sqrt{1-\sqrt{2-4E}} \tag{2.8}$$

where $E=E_x+E_y$ and $E_x=0.25$ is the separatrix energy. Now let ε be different from zero but sufficiently small. Then the hyperbolic set Λ is localized in the neighbourhood of the unperturbed separatrix solution. Trajectories which start with an impact parameter b near the value given by (2.8) are influenced by this hyperbolic set and we expect to find a discontinuous behaviour of the deflection function. However, (2.8) is a good approximation for very small ε only because it is based on the picture of two independent oscillators. A better estimation can be obtained by taking the interaction energy $H_w=\varepsilon x^2y^2$ of the two oscillators into account. The interaction reduces the energy part H_x+H_y which is available to the oscillators. At $x=1$, $y=b_0$ the interaction energy is given by $H_w(x=1, y=b_0)=\varepsilon b_0^2=\varepsilon(1-\sqrt{2-4E})$ where we have inserted the zero-order result (2.8). Taking this energy reduction into account, i.e. replacing in (2.8) E by $E-\varepsilon(1-\sqrt{2-4E})$ gives to first order in ε

$$b_1=b_0\left(1-\frac{\varepsilon}{1-b_0^2}\right). \tag{2.9}$$

Obviously, for b_0 near 1 (i.e. E near 0.5) this approximation breaks down. We discuss some numerical experiments to compare (2.8) and (2.9) in section 4.

3. Existence of elliptic and hyperbolic periodic orbits

The result of section 2 leaves open the question whether or not the chaotic scattering is fully developed. In order to study this question we will concentrate on the dynamics of periodic orbits under perturbation. The subharmonic Melnikov function is then used to study the bifurcation of smooth resonant tori for $\varepsilon=0$ into discrete periodic motions for $\varepsilon\neq0$ as described in [8, 13]. The unperturbed periodic solutions are

$$x(t)=\frac{\sqrt{2}k_1}{\sqrt{1+k_1^2}}\operatorname{sn}\left(\frac{t}{\sqrt{1+k_1^2}}, k_1\right) \qquad y(t)=\frac{\sqrt{2}k_2}{\sqrt{1+k_2^2}}\operatorname{sn}\left(\frac{t}{\sqrt{1+k_2^2}}, k_2\right) \tag{3.1}$$

where

$$k_1^2=\frac{1-\sqrt{1-4E_x}}{1+\sqrt{1-4E_x}} \qquad k_2^2=\frac{1-\sqrt{1-4E_y}}{1+\sqrt{1-4E_y}} \qquad 0\leqslant E_x, E_y<0.25. \tag{3.2}$$

E_x and E_y indicate the energy of the single oscillators (cf (2.4)). For a resonance of order m/n, we require

$$nT_1=mT_2 \tag{3.3}$$

where m, n are relatively prime natural numbers and $T_{1,2}$ are the periods of (3.1). The resonance condition relating the energies E_x and E_y is therefore

$$\frac{nK(k_1(E_x))}{\sqrt{1+\sqrt{1-4E_x}}}=\frac{mK(k_2(E_y))}{\sqrt{1+\sqrt{1-4E_y}}} \tag{3.4}$$

(cf (3.2)).

We work in a constant energy surface $H=E$ with $E\in(0,0.5)$. However, for the scattering process the interval $E\in(0.25,0.5)$ is important only. The resonance condition (3.4) has a unique solution for each pair (m,n) and this result implies that the unperturbed system has a dense set of resonant tori.

We next compute the subharmonic Melnikov function. Selecting a total energy E and an admissible pair (m,n), the Melnikov integral is given by (cf [13])

$$\begin{aligned} M(t_0;m,n,E) &= -\frac{T_2}{2\pi}\int_{-mT_2/2}^{mT_2/2} y^2(t+t_0)\frac{\mathrm{d}}{\mathrm{d}t}x^2(t)\,\mathrm{d}t \\ &= -\frac{2T_2k_1^2k_2^2}{\pi(1+k_1^2)(1+k_2^2)}\int_{-mT_2/2}^{mT_2/2}\mathrm{sn}^2\left(\frac{t+t_0}{\sqrt{1+k_2^2}},k_2\right)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}^2\left(\frac{t}{\sqrt{1+k_1^2}},k_1\right)\mathrm{d}t. \end{aligned} \tag{3.5}$$

Using the Fourier series expansion [14]

$$\mathrm{sn}^2(u,k)=\frac{\pi}{4k^3K^3}\sum_{i=0}^{\infty}\xi_i(k)\sin\left[(2i+1)\frac{\pi u}{2K}\right]$$

where

$$\xi_i(k)\equiv[4(1+k^2)K^2-(2i+1)^2\pi^2]\frac{q^{i+1/2}}{1-q^{2i+1}}$$

and $q=\exp(-\pi K'/K)$ is the elliptic nome, and interchanging the order of summation and integration, we have

$$\begin{aligned} M(t_0;m,n,E) &= A\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\xi_i(k_2)\xi_j(k_1)(2j+1) \\ &\quad\times\int_{-mT_2/2}^{mT_2/2}\sin\left[(2i+1)\frac{2\pi(t+t_0)}{T_2}\right]\cos\left[(2j+1)\frac{2\pi t}{T_1}\right]\mathrm{d}t \end{aligned}$$

where the constant A is defined by

$$A\equiv-\frac{T_2\pi^2}{4T_1(1+k_1^2)k_1K^3(k_1)(1+k_2^2)k_2K^3(k_2)}. \tag{3.6}$$

An application of the addition formulae yields

$$\begin{aligned} M(t_0;m,n,E) &= A\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\xi_i(k_2)\xi_j(k_1)(2j+1)\sin\left[(2i+1)\frac{2\pi t_0}{T_2}\right] \\ &\quad\times\int_{-mT_2/2}^{mT_2/2}\cos\left[(2i+1)\frac{2\pi t}{T_2}\right]\cos\left[(2j+1)\frac{2\pi t}{T_1}\right]\mathrm{d}t. \end{aligned} \tag{3.7}$$

Using the orthogonality of the Fourier components, one can see that the integral does not vanish if and only if

$$\frac{(2i+1)}{T_2}=\frac{(2j+1)}{T_1}\Leftrightarrow m(2i+1)=n(2j+1) \tag{3.8}$$

since $nT_1 = mT_2$. From condition (3.8) follows that m, n are both odd and one additionally has $(2i+1) = (2\mu+1)n, (2j+1) = (2\mu+1)m$ with $\mu = 0, 1, 2, \ldots$. Finally, one obtains

$$M(t_0; m, n, E) = \bar{A} \sum_{\mu=0}^{\infty} \frac{c_\mu(1)c_\mu(2)(2\mu+1)\sin\left[(2\mu+1)m\dfrac{2\pi t_0}{T_1}\right]}{\sinh\left[(2\mu+1)\dfrac{n\pi K'(k_2)}{2K(k_2)}\right]\sinh\left[(2\mu+1)\dfrac{m\pi K'(k_1)}{2K(k_1)}\right]} \tag{3.9}$$

where

$$\bar{A} \equiv \frac{-8\pi^2 m^2}{(T_1)^3 k_1 K(k_1) k_2 K(k_2)}$$

and

$$c_\mu(1) \equiv \left[\frac{T_1^2}{4} - (2\mu+1)^2 m^2 \pi^2\right] \qquad c_\mu(2) \equiv \left[\frac{T_2^2}{4} - (2\mu+1)^2 n^2 \pi^2\right].$$

Obviously, the series (3.9) converges uniformly in any t_0-interval. This follows from the fact that we have used the Fourier expansion of the continuous function $\mathrm{sn}^2(u, k)$. The same statement is valid for the series (2.6).

To count the number of periodic orbits for $\varepsilon \neq 0$ one has to count the number of zeros of the Melnikov function in a suitable t_0-interval. It is easy to show that the function (3.9) has at least $2m$ simple zeros in the interval $t_0 \in [0, T_1)$. This means that there are at least two isolated m-periodic orbits. For $E \in (0, 0.25)$ the pair (m, n) of odd, relatively prime integers is restricted to $m/n \in [K(0)/K(k)\sqrt{1+k^2}, K(k)\sqrt{1+k^2}/K(0)]$, where $k^2 = (1-2E-\sqrt{1-4E})/2E$ and for $E \in [0.25, 0.5)$ there is no restriction for (m, n). Therefore we obtain the following theorem (cf also [13]).

Theorem 2. For any integer $N < \infty$, there exists $\varepsilon(N) > 0$ such that, for $0 < \varepsilon \leqslant \varepsilon(N)$, on each energy surface $H = E \in (0, 0.5)$ of (2.2), and in any neighbourhood of an invariant torus for the unperturbed system there are at least N distinct periodic orbits.

Note that N must be finite because the Melnikov function (3.9) tends to zero as $m, n \to \infty$ and for $\varepsilon M \sim O(\varepsilon^2)$ the obtained first order perturbation results are becoming invalid. However, N can be chosen arbitrarily large if ε is sufficiently small.

Using the arguments of Arnold and Avez [15], it can be shown that half of the periodic orbits are hyperbolic and half are elliptic, if ε is small enough. For the scattering process with $E \in (0.25, 0.5)$ this means that the chaotic scattering arising from the heteroclinic intersections of section 2 is not fully developed. Additionally one finds the irrational KAM tori for ε sufficiently small. It must be underlined that in contrast to [13] the range of validity of theorem 2 contains also the scattering sector $E \in (0.25, 0.5)$.

4. Discussion and numerical experiments

In order to demonstrate the validity of the analytical predictions, some comparisons with numerical calculations have been carried out. Figure 4 shows parts of the stable

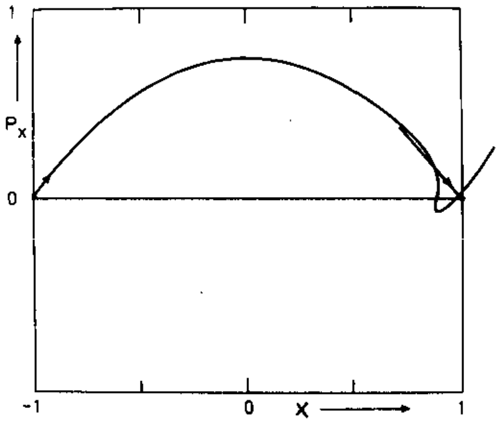

Figure 4. Parts of the unstable and stable manifolds of hyperbolic fixed points of the Poincaré map for $\varepsilon = 0.1$ and $E = 0.4$.

and unstable manifolds of the hyperbolic fixed points corresponding to the Poincaré map $P: \Sigma \to \Sigma$ for $\varepsilon = 0.1$. The surface of section Σ is defined as in section 2. The other branches of the manifolds (not shown here) can be obtained by using the symmetries of the equations of motion. Clearly visible is the predicted crossing of the stable and unstable manifolds. According to the heteroclinic theorem [7] there exists the invariant set Λ near the unperturbed separatrix which contains the bounded chaotic orbits.

Figure 5 shows the Poincaré section for selected initial values of the perturbed equations of motion. The total particle energy amounts to $E = 0.335$, and the existence of stable periodic orbits and KAM tori in the region which at $x = 0$ is approximately characterized by $\sqrt{2(E-0.25)} < p_x < \sqrt{0.5}$ can be understood as follows. For $\varepsilon = 0$ bounded motion of both oscillators is possible only in the mentioned interval of p_x. According to theorem 2 and the KAM theorem for sufficiently small ε we therefore observe periodic and quasiperiodic orbits near the unperturbed ones.

Theorems 1 and 2 make predictions only for transitions from the integrable to the non-integrable system, i.e. at $E =$ constant we vary the perturbation parameter ε. What is observed for $\varepsilon =$ constant as the energy E decreases from above the maximum height of the potential, $E_m = 1/2(1-2\varepsilon)$, to below? According to the results of Bleher *et al*

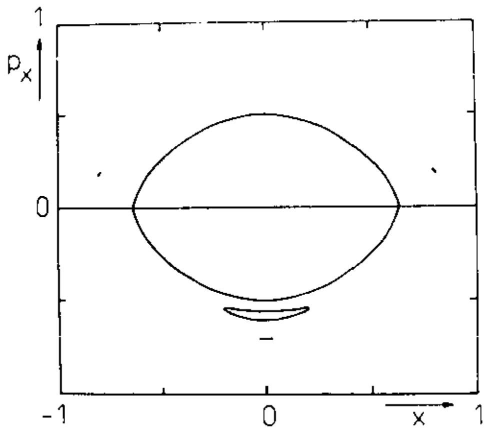

Figure 5. Poincaré section plot for selected initial data and $\varepsilon = 0.1$, $E = 0.335$.

[2] we expect an abrupt bifurcation to fully developed scattering. However, the numerical experiment shows in contrast to other scattering systems (see e.g. [2, 16]) that the energy range where hyperbolic chaos exists is relatively small and becomes even smaller as ε tends to zero. For example, for $\varepsilon = 0.5$ one has $E_m = 0.555$. Stable tori and stable periodic orbits can be observed until to $E \approx 0.528$. Above this value, i.e. in the interval $0.528 < E < E_m$, no stable bounded orbits are found in our numerical experiments. We suppose that in this range chaotic scattering is fully developed. However, we have no analytical proof of this result. The energy range $0.5 < E < E_m$ cannot be studied by using the Melnikov method because the unperturbed system has bounded solutions for $E \leqslant 0.5$ only.

In order to study the influence of the bounded invariant set Λ on the scattering behaviour we have performed some numerical calculations. We choose initial conditions outside of B, i.e. $x = 1.8$, $y = b$, $p_x = -\sqrt{2E}$, $p_y = 0$. In figure 6 the angle Θ between the outgoing particle velocity and the x axis is plotted as a function of the impact parameter b. For $\varepsilon = 0.1$ and $E = 0.45$ figure 6 clearly shows singularities in the deflection function $\Theta(b)$. Successive blow-ups (not shown here) suggest that the deflection function is singular on a Cantor set indicating the chaotic nature of the invariant set Λ. Of course, the position of the various intervals along the b axis depends upon E and ε. The dashed line in figure 6 shows $\Theta(b)$ in the integrable case $\varepsilon = 0$. Moreover, no singularities are observed for arbitrarily chosen values of the impact parameter.

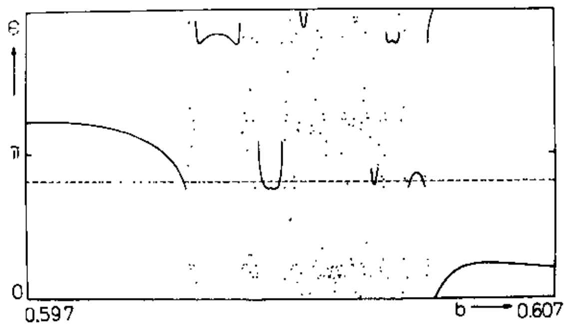

Figure 6. Representation of the deflection function $\Theta(b)$; $E = 0.45$, $\varepsilon = 0.1$, dashed line: $E = 0.45$, $\varepsilon = 0$.

Table 1 shows some results of numerical scattering experiments for selected values E and ε. The theoretical values of b_0 and b_1 are calculated by means of (2.8) and (2.9). The numerical value b_{num} is given by the position of the centre of the interval along the b axis in which the Cantor set of singularities is localized. The width of this interval depends upon ε and E and is relatively small in comparison with other scattering systems [1-3], e.g. for $\varepsilon = 0.1$ and $E = 0.4$ the width amounts to 0.006.

There is a good agreement between the numerical values b_{num} and the theoretical impact parameters b_1 for all energy values which are far from $E = 0.5$. According to the approximations in (2.9) the differences between b_{num} and b_1 become large for $E = 0.45$ and $\varepsilon > 0.1$. Our investigation shows that the Melnikov method can successfully be applied to near-integrable scattering systems. However, there are some further problems with regard to other important quantities of chaotic scattering. For example, the Melnikov function can be used to estimate escape rates, stretching coefficients and the topological entropy [17]. Moreover, the width of the stochastic layer, which is

Table 1. Critical values of the impact parameter (numerical results b_{num}, theoretical values b_0, b_1 according to (2.8) and (2.9)).

ε \ E		0.27	0.335	0.4	0.45
	b_0	0.202	0.433	0.606	0.743
0.01	b_1	0.200	0.427	0.596	0.726
	b_{num}	0.199	0.426	0.593	0.723
0.05	b_1	0.191	0.406	0.558	0.660
	b_{num}	0.190	0.400	0.552	0.660
0.1	b_1	0.181	0.379	0.510	0.577
	b_{num}	0.179	0.373	0.510	0.602
0.15	b_1	0.170	0.353	0.462	0.494
	b_{num}	0.170	0.351	0.476	0.558

described by the maximum of the Melnikov function, must be connected with the width of the irregular intervals of the deflection function.

Although we have investigated a particular system it is apparent that the same general approach can be used for other near-integrable scattering systems and we expect similar results with regard to the bifurcation behaviour.

Acknowledgments

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